

# TANGENTIAL EXTREMAL PRINCIPLES FOR FINITE AND INFINITE SYSTEMS OF SETS, II: APPLICATIONS TO SEMI-INFINITE AND MULTIOBJECTIVE OPTIMIZATION\*

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**Abstract.** This paper contains selected applications of the new tangential extremal principles and related results developed in [20] to calculus rules for infinite intersections of sets and optimality conditions for problems of semi-infinite programming and multiobjective optimization with countable constraints.

**Key words.** Variational analysis, extremal principles, semi-infinite programming, multiobjective optimization, generalized differentiation, tangent and normal cones, qualification conditions.

**AMS subject classifications.** Primary: 49J52, 49J53; Secondary: 90C30

## 1 Introduction

Variational analysis is based on variational principles and techniques, which are largely inspired and motivated by applications to constrained optimization and related problems. Extremal principles for systems of sets can be treated as variational principles in a geometric framework while playing a crucial role in the core variational theory and numerous applications; see, in particular, the books [5, 18, 19, 21, 22] and the references therein.

In [20], we developed new tangential extremal principles that concerned, for the first time in the literature, *countable* systems of sets. Our main motivation came from possible applications to problems of semi-infinite optimization with a countable number of constraints. It has been well recognized in optimization theory and its applications that problems of this type are significantly more difficult in comparison with conventional problems of semi-infinite optimization dealing with parameterized constraints over compact index sets; see, e.g., [15].

This paper mainly addresses selected applications of the tangential extremal principles and their consequences in [20] to various problems of semi-infinite optimization with countable constraints, particularly including those which naturally arise in semi-infinite programming and multiobjective optimization. To deal with such problems, we develop new calculus rules for tangent and normal cones to countable intersections of sets. These calculus results are certainly of their own interest being strongly used in the subsequent applications. To simplify the presentation, we confine ourselves to problems formulated in finite-dimensional spaces. At the same time, the initial data involved may be nonsmooth and nonconvex, and we strongly employ appropriate constructions of generalized differentiation in variational analysis.

The rest of the paper is organized as follows. Section 2 contains some preliminaries of variational analysis and also recall two major results from [20] largely used in the sequel.

Section 3 is devoted to calculus rules for tangent and normal cones to countable intersections of nonconvex sets and the corresponding qualification conditions. A special attention is paid in this section to a countable nonconvex version of the so-called “conical hull intersection property” (CHIP) developed earlier for finite intersections of convex sets and successfully used in convex

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optimization, approximation theory, etc. We establish verifiable sufficient conditions for the non-convex CHIP and employ this property and other qualification conditions to derive new calculus rules for generalized normals to infinite intersections of nonconvex sets in finite dimensions.

Section 4 presents a number of applications of the results from [20] and the from the preceding section to deriving necessary optimality conditions in various problems of semi-infinite programming with geometric, operator, and functional constraints. We obtain optimality conditions of different types under appropriate constraint qualifications and compare the optimality and qualification conditions obtained with those known before in convex and nonconvex settings.

Finally, Section 5 concerns applications of our major tangential extremal principle and the related calculus rules to various problems of multiobjective optimization including those with set-valued objectives. Besides paying the main attention to multiobjective problems with countable constraints, we introduce and develop there some notions and results, which seem to be of their own interest for the general theory of multiobjective optimization and its subsequent applications.

The notation and terminology of the paper are basically standard in variational analysis and generalized differentiation; cf. [20] and the books on variational analysis mentioned above. Recall that  $\mathbb{N} := \{1, 2, \dots\}$ , that  $B$  denotes the closed unit ball in  $\mathbb{R}^n$ , and that

$$\begin{aligned} \operatorname{Limsup}_{x \rightarrow \bar{x}} F(x) := \left\{ y \in \mathbb{R}^m \mid \exists \text{ sequences } x_k \rightarrow \bar{x} \text{ and } y_k \rightarrow y \right. \\ \left. \text{with } y_k \in F(x_k) \text{ for all } k \in \mathbb{N} \right\} \end{aligned} \quad (1.1)$$

stands for the (sequential) *Painlevé-Kuratowski upper/outer limit* of a set-valued mapping  $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  at a point  $\bar{x} \in \operatorname{dom} F := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$  of its domain.

## 2 Preliminaries from Variational Analysis

Our main references for the brief overview of this section are [18, 20, 21], where the reader can find proofs, discussions, and commentaries.

Given a set  $\Omega \subset \mathbb{R}^n$  locally closed around a point  $\bar{x} \in \Omega$ , we use in this paper the (only one) notion of the *tangent cone*  $T(\bar{x}; \Omega)$  given by

$$T(\bar{x}; \Omega) := \operatorname{Limsup}_{t \downarrow 0} \frac{\Omega - \bar{x}}{t}, \quad (2.1)$$

which is also known as the Bouligand-Severi contingent cone to  $\Omega$  at  $\bar{x}$ . The *normal cone*  $N(\bar{x}; \Omega)$  to  $\Omega$  at  $\bar{x}$  is defined by the outer limit (1.1) as

$$N(\bar{x}; \Omega) := \operatorname{Limsup}_{x \rightarrow \bar{x}} [\operatorname{cone}(x - \Pi(x; \Omega))] \quad (2.2)$$

via the Euclidean projection  $\Pi(x; \Omega) := \{w \in \Omega \mid \|x - w\| = \operatorname{dist}(x; \Omega)\}$  to  $\Omega$  at  $x \in \Omega$  and is known under that names of the Mordukhovich/basic/limiting normal cone to closed subsets of finite-dimensional spaces. Our basic normal cone (2.2) is often nonconvex while admitting the following outer limiting representation:

$$N(\bar{x}; \Omega) = \operatorname{Limsup}_{x \xrightarrow{\Omega} \bar{x}} \hat{N}(x; \Omega)$$

via the convex collections of *Fréchet normals* to  $\Omega$  at  $x \in \Omega$  given by

$$\widehat{N}(x; \Omega) := \left\{ x^* \in \mathbb{R}^n \mid \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq 0 \right\}, \quad (2.3)$$

where  $u \xrightarrow{\Omega} x$  means that  $u \rightarrow x$  with  $u \in \Omega$ . Note that  $\widehat{N}(\bar{x}; \Omega)$ , known also as the prenormal or regular normal cone, is dually generated by the (generally nonconvex) tangent cone (2.1) as

$$\widehat{N}(\bar{x}; \Omega) = T^*(\bar{x}; \Omega) := \{x^* \in \mathbb{R}^n \mid \langle x^*, v \rangle \leq 0 \text{ for all } v \in T(\bar{x}; \Omega)\}. \quad (2.4)$$

For convex sets  $\Omega$  all the constructions (2.1)–(2.3) reduce to the corresponding tangent and normal cones of convex analysis, while only the basic normal cone (2.2) enjoys comprehensive calculus rules (full calculus) in general nonconvex settings; see [18, 21] and their references. Note the following remarkable fact relating the tangent and normal cones to arbitrary closed sets  $\Omega \subset \mathbb{R}^n$  (see [21, Theorem 6.27] and [20, Corollary 6.5]):

$$N(0; T(\bar{x}; \Omega)) \subset N(\bar{x}; \Omega). \quad (2.5)$$

Given further a set-valued mapping  $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  with the graph

$$\text{gph } F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x)\},$$

we define the *coderivative* of  $F$  at  $(\bar{x}, \bar{y}) \in \text{gph } F$  via the normal cone (2.2) by

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph } F)\}, \quad y^* \in \mathbb{R}^m, \quad (2.6)$$

where  $\bar{y} = f(\bar{x})$  is omitted if  $F = f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is single-valued. Observe that the coderivative (2.6) is a positively homogeneous mapping  $D^*F(\bar{x}, \bar{y}): \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ , which reduces to the single-valued adjoint derivative operator

$$D^*f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\} \text{ for all } y^* \in \mathbb{R}^m \quad (2.7)$$

if  $f$  is *strictly differentiable* at  $\bar{x}$  in the sense that

$$\lim_{\substack{x \rightarrow \bar{x} \\ u \rightarrow \bar{x}}} \frac{f(x) - f(u) - \langle \nabla f(\bar{x}), x - u \rangle}{\|x - u\|} = 0;$$

the latter is automatic if  $f$  when  $C^1$  around  $\bar{x}$ .

Given finally an extended-real-valued function  $\varphi: \mathbb{R}^n \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$  finite at  $\bar{x}$ , we define its *basic subdifferential* at  $\bar{x}$  by

$$\partial\varphi(\bar{x}) := \{x^* \in \mathbb{R}^n \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi)\} \quad (2.8)$$

via the normal cone (2.2) to the epigraph  $\text{epi } \varphi := \{(x, \mu) \in \mathbb{R}^{n+1} \mid \mu \geq \varphi(x)\}$ . The subdifferential (2.8) can be equivalently represented as the outer limit

$$\partial\varphi(\bar{x}) = \text{Lim sup}_{x \xrightarrow{\varphi} \bar{x}} \widehat{\partial}\varphi(x),$$

with  $x \xrightarrow{\varphi} \bar{x}$  indicating that  $x \rightarrow \bar{x}$  and  $\varphi(x) \rightarrow \varphi(\bar{x})$ , of the Fréchet-like construction

$$\widehat{\partial}\varphi(x) := \{x^* \in \mathbb{R}^n \mid \liminf_{u \rightarrow x} \frac{\varphi(u) - \varphi(x) - \langle x^*, u - x \rangle}{\|u - x\|} \geq 0\}. \quad (2.9)$$

To conclude this section, we recall the concept of tangential extremality for countable set systems introduced in [20] and formulate two major results obtained therein, which are largely used in what follows. A set system  $\{\Omega_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n$  is *tangentially extremal* at  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$  if there is a bounded sequence  $\{a_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n$  such that

$$\bigcap_{i=1}^{\infty} [T(\bar{x}; \Omega_i) - a_i] = \emptyset. \quad (2.10)$$

**Theorem 2.1 (tangential extremal principle).** *Let a countable system  $\{\Omega_i\}_{i \in \mathbb{N}}$  of closed sets in  $\mathbb{R}^n$  be tangentially extremal at  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$ . Assume that*

$$\bigcap_{i=1}^{\infty} [T(x; \Omega_i)] = \{0\}. \quad (2.11)$$

*Then there are normal vectors*

$$x_i^* \in N(0; T(\bar{x}; \Omega_i)) \subset N(\bar{x}; \Omega_i) \text{ for all } i = 1, 2, \dots \quad (2.12)$$

*satisfying the following extremality conditions:*

$$\sum_{i=1}^{\infty} \frac{1}{2^i} x_i^* = 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{2^i} \|x_i^*\|^2 = 1. \quad (2.13)$$

The next result from [20] is based on the tangential extremal principle.

**Theorem 2.2 (representation of Fréchet normals to countable cone intersections).** *Let  $\{\Lambda_i\}_{i \in \mathbb{N}}$  be a countable system of closed cones in  $\mathbb{R}^n$  satisfying the conic qualification condition*

$$\left[ \sum_{i=1}^{\infty} x_i^* = 0, x_i^* \in N(0; \Lambda_i) \right] \implies [x_i^* = 0 \text{ for all } i \in \mathbb{N}]. \quad (2.14)$$

*Denoting the cone intersection by  $\Lambda := \bigcap_{i=1}^{\infty} \Lambda_i$ , we have the following representation of Fréchet normals to  $\Lambda$  at the origin:*

$$\widehat{N}(0; \Lambda) \subset \text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(0; \Lambda_i), I \in \mathcal{L} \right\}, \quad (2.15)$$

*where  $\mathcal{L}$  is the collection of all the finite subsets of  $\mathbb{N}$ .*

### 3 Tangents and Normals to Infinite Intersections of Sets

The main purpose of this section is to derive calculus rules for representing generalized normals to countable intersections of arbitrary closed sets under appropriate qualification conditions. Besides employing the tangential extremal principle, one of the major ingredients in our approach is relating calculus rules for generalized normals to countable set intersections with the so-called “conical hull intersection property” defined in terms of tangents to sets, which was intensively studied and applied in the literature for the case of finite intersections of convex sets; see, e.g., [4, 9, 10, 14, 17] and the references therein. In what follows, we keep the terminology of convex analysis (that goes back probably to [9]) replacing the tangent and normal cones therein by the nonconvex extension (2.1) and (2.2).

**Definition 3.1 (CHIP for countable intersections).** A set system  $\{\Omega_i\}_{i \in \mathbb{N}}$  in  $\mathbb{R}^n$  is said to have the CONICAL HULL INTERSECTION PROPERTY (CHIP) at  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$  if

$$T\left(\bar{x}; \bigcap_{i=1}^{\infty} \Omega_i\right) = \bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i). \quad (3.1)$$

In convex analysis and its applications the CHIP is often related to the so-called “strong CHIP” for finite set intersections expressed via the normal cone to the convex sets in question. Following this terminology in the case of infinite intersections of nonconvex sets, we say that a countable system of sets  $\{\Omega_i\}_{i \in \mathbb{N}}$  has the *strong conical hull intersection property* (or the *strong CHIP*) at  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$  if

$$N\left(\bar{x}; \bigcap_{i=1}^{\infty} \Omega_i\right) = \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(\bar{x}; \Omega_i), I \in \mathcal{L} \right\}. \quad (3.2)$$

When all the sets  $\Omega_i$  as  $i \in \mathbb{N}$  are *convex* in (3.2), the strong CHIP of the system  $\{\Omega_i\}_{i \in \mathbb{N}}$  can be equivalently written in the form

$$N\left(\bar{x}; \bigcap_{i=1}^{\infty} \Omega_i\right) = \text{co} \bigcup_{i=1}^{\infty} N(\bar{x}; \Omega_i). \quad (3.3)$$

We say that a countable set system  $\{\Omega_i\}_{i \in \mathbb{N}}$  has the *asymptotic strong CHIP* at  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$  if the latter representation is replaced by

$$N\left(\bar{x}; \bigcap_{i=1}^{\infty} \Omega_i\right) = \text{cl co} \bigcup_{i=1}^{\infty} N(\bar{x}; \Omega_i). \quad (3.4)$$

The next result shows the *equivalence* between the CHIP and the asymptotic strong CHIP for intersections of convex sets in finite dimensions. It follows from the proof that this equivalence holds for *arbitrary* intersections of convex sets, not only for countable ones studies in this paper.

**Theorem 3.2 (characterization of CHIP for intersections of convex sets).** Let  $\{\Omega_i\}_{i \in \mathbb{N}}$  be a system of convex sets in  $\mathbb{R}^n$ , and let  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$ . The following are equivalent:

- (a) The system  $\{\Omega_i\}_{i \in \mathbb{N}}$  has the CHIP at  $\bar{x}$ .
- (b) The system  $\{\Omega_i\}_{i \in \mathbb{N}}$  has the asymptotic strong CHIP at  $\bar{x}$ .

In particular, the strong CHIP implies the CHIP but not vice versa.

**Proof.** Observe first that for convex sets in finite dimensions, in addition to the duality property (2.4) with  $\widehat{N}(\bar{x}; \Omega)$  replaced by  $N(\bar{x}; \Omega)$ , we have the reverse duality representation

$$T(\bar{x}; \Omega) = N^*(\bar{x}; \Omega) := \{v \in \mathbb{R}^n \mid \langle x^*, v \rangle \leq 0 \text{ for all } x^* \in N(\bar{x}; \Omega)\}. \quad (3.5)$$

Let us now justify the equality

$$\left( \bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i) \right)^* = \text{cl co} \bigcup_{i=1}^{\infty} N(\bar{x}; \Omega_i). \quad (3.6)$$

The inclusion “ $\supset$ ” follows from (2.4) by the observation

$$N(\bar{x}; \Omega_i) = T^*(\bar{x}; \Omega_i) \subset \left( \bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i) \right)^*$$

due the closedness and convexity of the polar set on the right-hand side of the latter inclusion.

To prove the opposite inclusion “ $\subset$ ” in (3.6), pick some  $x^* \notin \text{cl co } \bigcup_{i=1}^{\infty} N(\bar{x}; \Omega_i)$ . Then the classical separation theorem for convex sets ensures the existence of a vector  $v \in \mathbb{R}^n$  such that

$$\langle x^*, v \rangle > 0 \quad \text{and} \quad \langle u^*, v \rangle \leq 0 \quad \text{for all } u^* \in \text{cl co } \bigcup_{i=1}^{\infty} N(\bar{x}; \Omega_i). \quad (3.7)$$

Hence for each  $i \in \mathbb{N}$  we get  $\langle u^*, v \rangle \leq 0$  whenever  $u^* \in N(\bar{x}; \Omega_i)$ , which implies that  $v \in N^*(\bar{x}; \Omega_i)$  and therefore  $v \in T(\bar{x}; \Omega_i)$  by (3.5). This gives us  $v \in \bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i)$ , and so  $x^* \notin \left( \bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i) \right)^*$  due to  $\langle x^*, v \rangle > 0$  in (3.7). Thus we get the inclusion “ $\subset$ ” in (3.6), which holds as equality. Similar arguments justify the fulfillment of the parallel duality relationship

$$\bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i) = \left( \text{cl co } \bigcup_{i=1}^{\infty} N(\bar{x}; \Omega_i) \right)^*. \quad (3.8)$$

Assuming now that the CHIP in (a) holds and employing (2.4) and (3.6) for the set intersection  $\Omega := \bigcap_{i=1}^{\infty} \Omega_i$ , we arrive at the equalities

$$N(\bar{x}; \Omega) = T^*(\bar{x}; \Omega) = \left( \bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i) \right)^* = \text{cl co } \bigcup_{i=1}^{\infty} N(\bar{x}; \Omega_i),$$

which give the asymptotic strong CHIP in (b). Conversely, assume that (b) holds. Then employing (3.5) and (3.8) implies the relationships

$$T(\bar{x}; \Omega) = N^*(\bar{x}; \Omega) = \left( \text{cl co } \bigcup_{i=1}^{\infty} N(\bar{x}; \Omega_i) \right)^* = \bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i),$$

which ensure the fulfillment of the CHIP in (a) and thus establish the equivalence the properties in (a) and (b). Since the strong CHIP implies the asymptotic strong CHIP due to the closedness of  $N(\bar{x}; \Omega)$ , it also implies the CHIP. The converse implication does not hold even for finitely many sets; counterexamples are presented, in particular, in [4, 14].  $\square$

The following simple consequence of Theorem 3.2 computes the normal cone to set of feasible solutions in linear semi-infinite programming with countable inequality constraints; cf. [7].

**Corollary 3.3 (normal cone to sets of feasible solutions of linear semi-infinite programs with countable constraints).** *Consider the set*

$$\Omega := \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \leq 0, \ i \in \mathbb{N}\}, \quad (3.9)$$

where the vectors  $a_i \in \mathbb{R}^n$  are fixed. Then the normal cone to  $\Omega$  at the origin is computed by

$$N(0; \Omega) = \text{cl co } \left[ \bigcup_{i=1}^{\infty} \{\lambda a_i \mid \lambda \geq 0\} \right]. \quad (3.10)$$

**Proof.** It is easy to see that the set (3.9) is represented as a countable intersection of sets having the CHIP. Furthermore, the asymptotic strong CHIP for this system is obviously (3.10). Thus the result follows immediately from Theorem 3.2.  $\square$

Let us show that the CHIP may be violated in rather simple situations involving finite and infinite intersections of convex sets defined by inequalities with convex (while nonlinear) functions.

**Example 3.4 (failure of CHIP for finite and infinite intersections of convex sets).**

(i) First consider the two convex sets

$$\Omega_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq x_1^2\} \quad \text{and} \quad \Omega_2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq -x_1^2\}$$

and their intersection at  $\bar{x} = (0, 0)$ . We have

$$\Omega_1 \cap \Omega_2 = \{\bar{x}\}, \quad T(\bar{x}; \Omega_1) = \mathbb{R} \times \mathbb{R}_+, \quad \text{and} \quad T(\bar{x}; \Omega_2) = \mathbb{R} \times \mathbb{R}_-.$$

Thus the CHIP does not hold in this case, since

$$T(\bar{x}; \Omega_1 \cap \Omega_2) = \{(0, 0)\} \neq T(\bar{x}; \Omega_1) \cap T(\bar{x}; \Omega_2) = \mathbb{R} \times \{0\}.$$

(ii) In the next case we have the CHIP violation for the countable intersection of convex sets, with the intersection set having nonempty interior. For each  $i \in \mathbb{N}$ , define  $\varphi_i(x) := ix^2$  if  $x < 0$  and  $\varphi_i(x) := 0$  if  $x \geq 0$ . Let  $\Omega_i := \text{epi } \varphi_i$  and  $\bar{x} = (0, 0)$ . It is easy to see that

$$\bigcap_{i=1}^{\infty} \Omega_i = \mathbb{R}_+ \times \mathbb{R}_+ \quad \text{and} \quad T(\bar{x}, \Omega_i) = \mathbb{R} \times \mathbb{R}_+ \quad \text{for } i \in \mathbb{N}.$$

It gives therefore the relationships

$$T\left(\bar{x}, \bigcap_{i=1}^{\infty} \Omega_i\right) = \mathbb{R}_+ \times \mathbb{R}_+ \neq \bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i) = \mathbb{R} \times \mathbb{R}_+, \quad i \in \mathbb{N},$$

which show that the CHIP fails for this system of sets at the origin.

Of course, we cannot expect to extend the equivalence of Theorem 3.2 to intersections of nonconvex sets. In what follows we are mainly interested in obtaining calculus rules for generalized normals (i.e., to get results of the “strong CHIP” type) using the nonconvex CHIP from Definition 3.1 (i.e., a calculus rule for tangents) as an appropriate assumption together with additional qualification conditions. Observe that the implication  $\text{CHIP} \implies \text{strong CHIP}$  does hold even for finite intersections of convex sets; see Theorem 3.2.

To implement this strategy, we first intend to obtain some sufficient conditions for the CHIP of countable intersections of nonconvex sets. Note that a number of sufficient conditions for the CHIP has been proposed for finite intersections of convex sets, where convex interpolation techniques play a particularly important role; see [4, 9, 10, 17] and the references therein. However, such techniques do not seem to be useful in nonconvex settings. To proceed in deriving sufficient conditions for the CHIP of countable nonconvex intersections, we explore some other possibilities.

Let us start with extending the concept and techniques of linear regularity in the direction of [4, 17, 23] to the case of infinite nonconvex systems; cf. various results and discussions therein on

particular cases of linear regularity and its applications. Given a countable system of closed sets  $\{\Omega_i\}_{i \in \mathbb{N}}$ , we say that it is *linearly regular* at  $\bar{x} \in \Omega := \bigcap_{i=1}^{\infty} \Omega_i$  if there exist a neighborhood  $U$  of  $\bar{x}$  and a positive number  $C > 0$  such that

$$\text{dist}(x; \Omega) \leq C \sup_{i \in \mathbb{N}} \{\text{dist}(x; \Omega_i)\} \quad \text{for all } x \in U. \quad (3.11)$$

In the next proposition we denote for convenience the distance function  $\text{dist}(x; \Omega)$  by  $d_{\Omega}(x)$  and employ the standard notion of equi-convergence for families of functions.

**Proposition 3.5 (sufficient conditions for CHIP in terms of linear regularity).** *Let  $\{\Omega_i\}_{i \in \mathbb{N}}$  be a countable system of closed sets in  $\mathbb{R}^n$  with the intersection  $\Omega := \bigcap_{i=1}^{\infty} \Omega_i$ , and let  $\bar{x} \in \Omega$ . Assume that the system of sets  $\{\Omega_i\}_{i \in \mathbb{N}}$  is linearly regular at  $\bar{x}$  with some  $C > 0$  in (3.11) and that the family of functions  $\{d_{\Omega_i}(\cdot)\}_{i \in \mathbb{N}}$  is equi-directionally differentiable at  $\bar{x}$  in the sense that for any  $h \in \mathbb{R}^n$  the functions*

$$\left\{ \frac{d_{\Omega_i}(\bar{x} + th)}{t}, i \in \mathbb{N} \right\}$$

*equi-converge as  $t \downarrow 0$  to the corresponding directional derivatives  $d'_{\Omega_i}(\bar{x}; h)$  uniformly in  $i \in \mathbb{N}$ . Then for all  $h \in \mathbb{R}^n$  and the positive constant  $C$  from (3.11) we have the estimate*

$$\text{dist}(h; \Lambda) \leq C \sup_{i \in \mathbb{N}} \{\text{dist}(h; \Lambda_i)\} \quad \text{with } \Lambda := T(\bar{x}; \Omega) \quad \text{and } \Lambda_i := T(\bar{x}; \Omega_i) \quad \text{as } i \in \mathbb{N}. \quad (3.12)$$

*In particular, the set system  $\{\Omega_i\}_{i \in \mathbb{N}}$  satisfies the CHIP at  $\bar{x}$ .*

**Proof.** Fixing  $h \in \mathbb{R}^n$  and using definition (2.1) of the tangent cone, we get

$$\text{dist}(h; \Lambda) = \liminf_{t \downarrow 0} \text{dist}\left(h; \frac{\Omega - \bar{x}}{t}\right) = \liminf_{t \downarrow 0} \frac{\text{dist}(\bar{x} + th; \Omega)}{t}.$$

When  $t$  is small, by the assumed linear regularity yields that

$$\frac{\text{dist}(\bar{x} + th; \Omega)}{t} \leq C \sup_{i \in \mathbb{N}} \frac{\text{dist}(\bar{x} + th; \Omega_i)}{t}.$$

Applying [6, Theorem 4] with the assumption of equi-directional differentiability, we have

$$\frac{\text{dist}(\bar{x} + th; \Omega_i)}{t} \rightarrow d'_{\Omega_i}(\bar{x}; h) = \text{dist}(h; \Lambda_i) \quad \text{uniformly in } i \text{ as } t \downarrow 0,$$

i.e., for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that whenever  $t \in (0, \delta)$  we have

$$\left\| \frac{\text{dist}(\bar{x} + th; \Omega_i)}{t} - \text{dist}(h; \Lambda_i) \right\| \leq \varepsilon \quad \text{for all } i \in \mathbb{N}.$$

Hence it holds for any  $t \in (0, \delta)$  that

$$\sup_{i \in \mathbb{N}} \frac{\text{dist}(\bar{x} + th; \Omega_i)}{t} \leq \sup_{i \in \mathbb{N}} \{\text{dist}(h; \Lambda_i)\} + \varepsilon.$$

Combining all the above, we get the estimates

$$\text{dist}(h; \Lambda) \leq C \liminf_{t \downarrow 0} \sup_{i \in \mathbb{N}} \frac{\text{dist}(\bar{x} + th; \Omega_i)}{t} \leq C \sup_{i \in \mathbb{N}} \{\text{dist}(h; \Lambda_i)\} + C\varepsilon,$$



which imply (3.12), since  $\varepsilon$  was chosen arbitrarily. Finally, the CHIP of the system  $\{\Omega_i\}_{i \in \mathbb{N}}$  at  $\bar{x}$  follows directly from (3.12) and the definitions.  $\square$

Now we present a consequence of Proposition 3.5 that simplifies the verification of linear regularity for countable set systems.

**Corollary 3.6 (CHIP via simplified linear regularity).** *Let  $\{\Omega_i\}_{i \in \mathbb{N}}$  be a countable system of closed subsets in  $\mathbb{R}^n$ , and let  $\bar{x} \in \Omega = \bigcap_{i=1}^{\infty} \Omega_i$ . Assume that the family  $\{d(\cdot; \Omega_i)\}_{i \in \mathbb{N}}$  is equi-directionally differentiable at  $\bar{x}$  and that there are numbers  $C > 0$ ,  $j \in \mathbb{N}$ , and a neighborhood  $U$  of  $\bar{x}$  such that we have the estimate*

$$\text{dist}(x; \Omega) \leq C \sup_{i \neq j} \{\text{dist}(x; \Omega_i)\} \quad \text{for all } x \in \Omega_j \cap U.$$

*Then the set system  $\{\Omega_i\}_{i \in \mathbb{N}}$  satisfies the CHIP at  $\bar{x}$ .*

**Proof.** Employing Proposition 3.5, it suffices to show that the set system  $\{\Omega_i\}_{i \in \mathbb{N}}$  is linearly regular at  $\bar{x}$ . To proceed, take  $r > 0$  so small that

$$\text{dist}(x; \Omega) \leq C \sup_{i \neq j} \{\text{dist}(x; \Omega_i)\} \quad \text{for all } x \in \Omega_j \cap (\bar{x} + 3r\mathcal{B}).$$

Since the distance function is nonexpansive, for every  $y \in \Omega_j \cap (\bar{x} + 3r\mathcal{B})$  and  $x \in \mathbb{R}^n$  we have

$$\begin{aligned} 0 &\leq C \sup_{i \neq j} \{\text{dist}(y; \Omega_i)\} - \text{dist}(y; \Omega) \leq C \sup_{i \neq j} \left( \{\text{dist}(x; \Omega_i)\} + \|x - y\| \right) - \text{dist}(x; \Omega) + \|x - y\| \\ &\leq C \sup_{i \neq j} \{\text{dist}(x; \Omega_i)\} - \text{dist}(x; \Omega) + (C + 1)\|x - y\|. \end{aligned}$$

Then it follows for all  $x \in \mathbb{R}^n$  that

$$\text{dist}(x; \Omega) \leq (2C + 1) \max \left[ \sup_{i \neq j} \{\text{dist}(x; \Omega_i)\}, \text{dist}(x; \Omega_j \cap (\bar{x} + 3r\mathcal{B})) \right].$$

Thus the linear regularity of  $\{\Omega_i\}_{i \in \mathbb{N}}$  at  $\bar{x}$  in the form of

$$\text{dist}(x; \Omega) \leq (2C + 1) \sup_{i \in \mathbb{N}} \{\text{dist}(x; \Omega_i)\}$$

would follow now from the relationship

$$\text{dist}(x; \Omega_j \cap (\bar{x} + 3r\mathcal{B})) = \text{dist}(x; \Omega_j) \quad \text{for all } x \in (\bar{x} + r\mathcal{B}). \quad (3.13)$$

To show (3.13), fix a vector  $x \in (\bar{x} + r\mathcal{B})$  above and pick any  $y \in \Omega_j \setminus (\bar{x} + 3r\mathcal{B})$ . This readily gives us  $\|x - y\| \geq \|y - \bar{x}\| - \|\bar{x} - x\| \geq 3r - r = 2r$  and implies that

$$\text{dist}(x; \Omega_j \setminus (\bar{x} + 3r\mathcal{B})) \geq 2r \quad \text{while} \quad \text{dist}(x; \Omega_j \cap (\bar{x} + 3r\mathcal{B})) \leq \|x - \bar{x}\| \leq r.$$

Hence we get the equalities

$$\text{dist}(x; \Omega_j) = \min \left\{ \text{dist}(x; \Omega_j \setminus (\bar{x} + 3r\mathcal{B})), \text{dist}(x; \Omega_j \cap (\bar{x} + 3r\mathcal{B})) \right\} = \text{dist}(x; \Omega_j \cap (\bar{x} + 3r\mathcal{B})),$$

which justify (3.13) and thus complete the proof of the corollary.  $\square$

The next proposition, which holds in fact for arbitrary (not only countable) intersections of sets, establishes a new sufficient condition for the CHIP of  $\{\Omega_i\}_{i \in \mathbb{N}}$ . To formulate it, we introduce a notion of the *tangential rank* of the intersection  $\Omega := \bigcap_{i=1}^{\infty} \Omega_i$  at  $\bar{x} \in \Omega$  by

$$\rho_{\Omega}(\bar{x}) := \inf_{i \in \mathbb{N}} \left\{ \limsup_{\substack{x \rightarrow \bar{x} \\ x \in \Omega_i \setminus \{\bar{x}\}}} \frac{\text{dist}(x; \Omega)}{\|x - \bar{x}\|} \right\}, \quad (3.14)$$

where we put  $\rho_{\Omega}(\bar{x}) := 0$  if  $\Omega_i = \{\bar{x}\}$  for at least one  $i \in \mathbb{N}$ .

**Proposition 3.7 (sufficient condition for CHIP via tangential rank of intersection).**

*Given a countable system of closed sets  $\{\Omega_i\}_{i \in \mathbb{N}}$  in  $\mathbb{R}^n$ , suppose that  $\rho_{\Omega}(\bar{x}) = 0$  for the tangential rank of their intersection  $\Omega := \bigcap_{i=1}^{\infty} \Omega_i$  at  $\bar{x} \in \Omega$ . Then this system exhibits the CHIP at  $\bar{x}$ .*

**Proof.** The result holds trivially if  $\Omega_i = \{\bar{x}\}$  for some  $i \in \mathbb{N}$ . Assume that  $\Omega_i \setminus \{\bar{x}\} \neq \emptyset$  for all  $i \in \mathbb{N}$  and observe that  $T(\bar{x}; \Omega) \subset T(\bar{x}; \Omega_i)$  whenever  $i \in \mathbb{N}$ . Thus we have

$$T(\bar{x}; \Omega) \subset \bigcap_{i \in \mathbb{N}} T(\bar{x}; \Omega_i).$$

To prove the reverse inclusion, fix an arbitrary vector  $0 \neq v \in \bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i)$ . By the assumption of  $\rho_{\Omega}(\bar{x}) = 0$  and definition (3.14), for any  $k \in \mathbb{N}$  we find a set  $\Omega_k$  from the system under consideration such that

$$\limsup_{\substack{x \rightarrow \bar{x} \\ x \in \Omega_k \setminus \{\bar{x}\}}} \frac{\text{dist}(x; \Omega)}{\|x - \bar{x}\|} < \frac{1}{k}.$$

Since  $v \in T(\bar{x}; \Omega_k)$ , there are sequences  $\{x_j\}_{j \in \mathbb{N}} \subset \Omega_k$  and  $t_j \downarrow 0$  satisfying

$$x_j \rightarrow \bar{x} \quad \text{and} \quad \frac{x_j - \bar{x}}{t_j} \rightarrow v \quad \text{as } j \rightarrow \infty,$$

which in turn implies the limiting estimate

$$\limsup_{j \rightarrow \infty} \frac{\text{dist}(x_j; \Omega)}{\|x_j - \bar{x}\|} < \frac{1}{k}.$$

The latter allows us to find a vector  $x_k \in \{x_j\}_{j \in \mathbb{N}}$  with  $\|x_k - \bar{x}\| \leq 1/k$  and the corresponding number  $t_k \leq 1/k$  such that

$$\left\| \frac{x_k - \bar{x}}{t_k} - v \right\| \leq \frac{1}{k} \quad \text{and} \quad \frac{\text{dist}(x_k; \Omega)}{\|x_k - \bar{x}\|} < \frac{1}{k}.$$

Then it follows that there exists  $z_k \in \Omega$  satisfying the relationships

$$\|z_k - x_k\| < \frac{1}{k} \|x_k - \bar{x}\| \leq \frac{1}{k^2}.$$

Combining all the above together gives us the estimates

$$\left\| \frac{z_k - \bar{x}}{t_k} - v \right\| \leq \left\| \frac{z_k - x_k}{t_k} \right\| + \left\| \frac{x_k - \bar{x}}{t_k} - v \right\| \leq \frac{1}{k} \left\| \frac{x_k - \bar{x}}{t_k} \right\| + \frac{1}{k} \leq \frac{1}{k} \left( \|v\| + \frac{1}{k} \right) + \frac{1}{k}, \quad k \in \mathbb{N}.$$

Now letting  $k \rightarrow \infty$ , we get  $z_k \xrightarrow{\Omega} \bar{x}$ ,  $t_k \downarrow 0$ , and  $\left\| \frac{z_k - \bar{x}}{t_k} - v \right\| \rightarrow 0$ . The latter verifies that  $v \in T(\bar{x}; \Omega)$  and thus completes the proof of the proposition.  $\square$

To conclude our discussions on the CHIP, we give yet another verifiable condition ensuring the fulfillment of this property for countable intersections of closed sets. We say that a set  $A$  is of *invex type* if it can be represented as the complement to a union with respect to  $t \in T$  of some open convex sets  $A_t$ , i.e.,

$$A = \mathbb{R}^n \setminus \bigcup_{t \in T} A_t, \quad (3.15)$$

The following lemma needed for the next proposition is also used in Section 5.

**Lemma 3.8 (sets of invex type).** *Let  $A \subset \mathbb{R}^n$  be a set of invex type, and let  $\bar{x} \in \bigcap_{t \in T} \text{bd } A_t \cap \text{bd } A$  be taken from the boundary intersections. Then we have the inclusion involving the tangent cone  $T(\bar{x}; A)$  to  $A$  at  $\bar{x}$ :*

$$\bar{x} + T(\bar{x}; A) \subset A. \quad (3.16)$$

**Proof.** To justify inclusion (3.16), suppose on the contrary that there is  $v \in T(\bar{x}; A)$  such that  $\bar{x} + v \notin A$ . For this vector  $v$  we find by definition (2.1) sequences  $s_k \downarrow 0$  and  $x_k \in A$  such that  $\frac{x_k - \bar{x}}{s_k} \rightarrow v$  as  $k \rightarrow \infty$ . Since  $\bar{x} + v \notin A$ , by invexity (3.15) there exists an index  $t \in T$  for which  $\bar{x} + v \in A_t$ . Thus we get the inclusion

$$\bar{x} + \frac{x_k - \bar{x}}{s_k} \in A_t \text{ for all } k \in \mathbb{N} \text{ sufficiently large.}$$

Then employing the convexity of  $A_t$  gives us that

$$x_k = (1 - s_k)\bar{x} + s_k\left(\bar{x} + \frac{x_k - \bar{x}}{s_k}\right) \in A_t$$

for the fixed index  $t \in T$  and all large numbers  $k \in \mathbb{N}$ . This contradicts the choice of  $x_k \in A$  and thus justifies the claimed inclusion (3.16).  $\square$

Now we are ready to derive the aforementioned sufficient condition for the CHIP.

**Proposition 3.9 (CHIP for countable intersections of invex-type sets).** *Given a countable system  $\{\Omega_i\}_{i \in \mathbb{N}}$  in  $\mathbb{R}^n$ , assume that there is a (possibly infinite) index subset  $J \subset \mathbb{N}$  such that each  $\Omega_i$  for  $i \in J$  is the complement to an open and convex set in  $\mathbb{R}^n$  and that*

$$\bar{x} \in \bigcap_{i \in J} \text{bd } \Omega_i \cap \text{int } \bigcap_{i \notin J} \Omega_i \quad (3.17)$$

for some  $\bar{x}$ . Then the system  $\{\Omega_i\}_{i \in \mathbb{N}}$  enjoys the CHIP at  $\bar{x}$ .

**Proof.** Take any  $\Omega_i$  with  $i \in J$  and find a convex and open set  $A \subset \mathbb{R}^n$  such that  $\Omega = \mathbb{R}^n \setminus A$ . Then  $\bar{x} \in \text{bd } A \cap \text{bd } \Omega_i$  by (3.17). Then Lemma 3.8 ensures that  $\bar{x} + T(\bar{x}; \Omega_i) \subset \Omega_i$  for this index  $i \in J$ . By the choice of  $\bar{x}$  in (3.17) we have furthermore that

$$\bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i) = \bigcap_{i \in J} T(\bar{x}; \Omega_i) \subset \bigcap_{i \in J} (\Omega_i - \bar{x}).$$

Since the set on the left-hand side of the latter inclusion is a cone, it follows that

$$\bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i) \subset T\left(0; \bigcap_{i \in J} (\Omega_i - \bar{x})\right) = T\left(\bar{x}; \bigcap_{i \in J} \Omega_i\right) = T\left(\bar{x}; \bigcap_{i=1}^{\infty} \Omega_i\right). \quad (3.18)$$

As the opposite inclusion in (3.18) is obvious, we conclude that the CHIP is satisfied for the countable set system  $\{\Omega_i\}_{i \in \mathbb{N}}$  at  $\bar{x}$ .  $\square$

In the last part of this section we show that the CHIP for countable intersections of nonconvex sets, combined with some other classification conditions, allows us to derive principal calculus rules for representing *generalized normals to infinite set intersections*. Thus the verifiable sufficient conditions for the CHIP established above largely contribute to the implementation of these calculus rules. Note that the results obtained in this direction provide new information even for convex set intersections, since in this case they furnish the required implication  $\text{CHIP} \implies \text{strong CHIP}$ , which does not hold in general nonconvex settings; see Theorem 3.2 for more discussions.

First we formulate and discuss appropriate qualification conditions for countable systems of sets in terms of the basic normal cone (2.2).

**Definition 3.10 (normal closedness and qualification conditions for countable set systems).** Let  $\{\Omega_i\}_{i \in \mathbb{N}}$  be a countable system of sets, and let  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$ . We say that:

(a) The set system  $\{\Omega_i\}_{i \in \mathbb{N}}$  satisfies the NORMAL CLOSEDNESS CONDITION (NCC) at  $\bar{x}$  if the combination of basic normals

$$\left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(\bar{x}; \Omega_i), I \in \mathcal{L} \right\} \text{ is closed in } \mathbb{R}^n, \quad (3.19)$$

where  $\mathcal{L}$  stands for the collection of all the finite subsets of  $\mathbb{N}$ .

(b) The system  $\{\Omega_i\}_{i \in \mathbb{N}}$  satisfies the NORMAL QUALIFICATION CONDITION (NQC) at  $\bar{x}$  if the following implication holds:

$$\left[ \sum_{i=1}^{\infty} x_i^* = 0, \ x_i^* \in N(\bar{x}; \Omega_i) \right] \implies \left[ x_i^* = 0 \text{ for all } i \in \mathbb{N} \right]. \quad (3.20)$$

The NCC in Definition 3.10(a) relates to various versions of the so-called *Farkas-Minkowski qualification condition* and its extensions for finite and infinite systems of sets. We refer the reader to, e.g., [12, 13] and the bibliographies therein, as well as to subsequent discussions in Section 4, for a number of results in this direction concerning convex infinite inequality systems and to [8] for more details on linear inequality systems with arbitrary index sets in general Banach spaces.

The NQC in Definition 3.10(b) is a direct extension of the corresponding condition (2.14) for system of cones. The counterpart of (3.20) for finite systems of sets is studied and applied in [18, 19] under the same name. The following proposition presents a simple sufficient condition for the validity of the NQC in the case of countable systems of convex sets.

**Proposition 3.11 (NQC for countable systems of convex sets).** Let  $\{\Omega_i\}_{i \in \mathbb{N}}$  be a system of convex sets for which there is an index  $i_0 \in \mathbb{N}$  such that

$$\Omega_{i_0} \cap \bigcap_{i \neq i_0} \text{int } \Omega_i \neq \emptyset. \quad (3.21)$$

Then the NQC in (3.20) is satisfied for the system  $\{\Omega_i\}_{i \in \mathbb{N}}$  at any  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$ .

**Proof.** Suppose without loss of generality that  $i_0 = 1$  and fix some  $w \in \Omega_1 \cap \bigcap_{i=2}^{\infty} \text{int } \Omega_i$ . Taking any normals  $x_i^* \in N(\bar{x}; \Omega_i)$  as  $i \in \mathbb{N}$  satisfying

$$\sum_{i=1}^{\infty} x_i^* = 0,$$

we get by the convexity of the sets  $\Omega_i$  that  $\langle x_i^*, w - \bar{x} \rangle \leq 0$  for all  $i \in \mathbb{N}$ . Then it follows that

$$\langle x_i^*, w - \bar{x} \rangle = - \sum_{j \neq i} \langle x_j^*, w - \bar{x} \rangle \geq 0, \quad i \in \mathbb{N},$$

which yields  $\langle x_i^*, w - \bar{x} \rangle = 0$  whenever  $i \in \mathbb{N}$ . Next fix  $\varepsilon > 0$  and find  $m \in \mathbb{N}$  so large that

$$\left\| \sum_{i=m+1}^{\infty} x_i^* \right\| \leq \varepsilon.$$

Pick  $u \in \mathbb{R}^n$  with  $\|u\| = 1$  and taking into account that  $w \in \bigcap_{i=2}^m \text{int } \Omega_i$ , we get

$$\lambda \langle x_i^*, u \rangle = \langle x_i^*, w + \lambda u - \bar{x} \rangle \leq 0, \quad i = 2, 3, \dots,$$

whenever  $\lambda > 0$  is sufficiently small. This implies that

$$\lambda \langle x_1^*, u \rangle = -\lambda \sum_{i=2}^m \langle x_i^*, u \rangle - \lambda \sum_{i=m+1}^{\infty} \langle x_i^*, u \rangle \geq -\lambda \left\| \sum_{i=m+1}^{\infty} x_i^* \right\| \cdot \|u\| \geq -\lambda \varepsilon,$$

which gives  $\langle x_1^*, u \rangle \geq -\varepsilon$ . Since  $\varepsilon > 0$  was chosen arbitrarily, we conclude that  $\langle x_1^*, u \rangle \geq 0$ . Repeating the same procedure for  $-u$  shows that  $\langle x_1^*, -u \rangle \geq 0$  and so  $\langle x_1^*, u \rangle = 0$  for all  $u \in \mathbb{R}^n$  with  $\|u\| = 1$ . This implies that  $x_1^* = 0$ . The same procedure ensures that  $x_i^* = 0$  for all  $i \in \mathbb{N}$ , which completes the proof of the proposition.  $\square$

Finally, we obtain the main result of this section, which expresses Fréchet normal to infinite set intersections via basic normals to the sets involved under the above CHIP and qualification conditions. This major calculus rule for arbitrary closed sets employs the corresponding intersection rule for cones from Theorem 2.2, which is based on the tangential extremal principle.

**Theorem 3.12 (generalized normals to countable set intersections).** *Let  $\{\Omega_i\}_{i \in \mathbb{N}}$  be a countable system of closed sets in  $\mathbb{R}^n$ , and let  $\bar{x} \in \Omega := \bigcap_{i=1}^{\infty} \Omega_i$ . Assume that the CHIP in (3.1) and NQC in (3.20) are satisfied for  $\{\Omega_i\}_{i \in \mathbb{N}}$  at  $\bar{x}$ . Then we have the inclusion*

$$\widehat{N}(\bar{x}; \Omega) \subset \text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(\bar{x}; \Omega_i), I \in \mathcal{L} \right\}, \quad (3.22)$$

where  $\mathcal{L}$  stands for the collection of all the finite subsets of  $\mathbb{N}$ . If in addition the CQC in (3.19) holds for  $\{\Omega_i\}_{i \in \mathbb{N}}$  at  $\bar{x}$ , then the closure operation can be omitted on the right-hand side of (3.22).

**Proof.** Using the assumed CHIP for  $\{\Omega_i\}_{i \in \mathbb{N}}$  at  $\bar{x}$ , constructions (2.1) and (2.3), and the duality correspondence (2.4) gives us

$$\widehat{N}(\bar{x}; \Omega) = \widehat{N}(0; T(\bar{x}; \Omega)) = \widehat{N}\left(0; \bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i)\right). \quad (3.23)$$

It follows from (2.5) that  $N(0; T(\bar{x}; \Omega_i)) \subset N(\bar{x}; \Omega_i)$  for all  $i \in \mathbb{N}$ , and thus the assumed NQC in (3.20) implies the conic one in (2.14). Applying Theorem 2.2, we have

$$\widehat{N}\left(0; \bigcap_{i=1}^{\infty} T(\bar{x}; \Omega_i)\right) \subset \text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(0; T(\bar{x}; \Omega_i)), I \in \mathcal{L} \right\}.$$

Now the intersection rule (3.22) follows from (2.5) and (3.23). Finally, the closure operation in (3.22) can be obviously dropped if the system  $\{\Omega_i\}_{i \in \mathbb{N}}$  satisfies the CQC at  $\bar{x}$ .  $\square$

## 4 Applications to Semi-Infinite Programming

This section is devoted to deriving necessary optimality conditions for various problems of semi-infinite programming (SIP) with *countable* constraints. As mentioned in Section 1, problems with countable constraints are among the most difficult in SIP, in comparison with conventional ones involving constraints indexed by compact sets. In fact, SIP problems with countable constraints are not different from seemingly more general problems with arbitrary index sets. Problems of the latter class have drawn particular attention in a number of recent publications, where some special structures of this type (mostly with linear and convex inequality constraints) have been considered; see, e.g., [8, 12, 13] and the references therein. In this section we derive, based on the tangential extremal principle and its calculus consequences, new optimality conditions for SIP with various types of countable constraints and compare them with those known in the literature.

Let us start with SIP involving countable constraints of the *geometric type*:

$$\text{minimize } \varphi(x) \text{ subject to } x \in \Omega_i \text{ as } i \in \mathbb{N}, \quad (4.1)$$

where  $\varphi: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is an extended-real-valued function, and where  $\{\Omega_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n$  is a countable system of constraint sets. Considering in general problems with nonsmooth and nonconvex cost functions and following the classification of [19, Chapter 5], we derive necessary optimality conditions of two kinds for (4.1) and other SIP *minimization* problems: *lower subdifferential* and *upper subdifferential* ones. Conditions of the “lower” kind are more conventional for minimization dealing with usual (lower) subdifferential constructions. On the other hand, conditions of the “upper” kind employ upper subdifferential (or superdifferential) constructions, which seem to be more appropriate for maximization problems while bringing significantly stronger information for special classes of minimizing cost functions in comparison with lower subdifferential ones; see [19] for more discussions, examples, and references.

We begin with upper subdifferential optimality conditions for (4.1). Given  $\varphi: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  finite at  $\bar{x}$ , the *upper subdifferential* of  $\varphi$  at  $\bar{x}$  used in this paper is of the Fréchet type defined by

$$\widehat{\partial}^+ \varphi(\bar{x}) := -\widehat{\partial}(-\varphi)(\bar{x}) = \left\{ x^* \in \mathbb{R}^n \mid \limsup_{x \rightarrow \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\} \quad (4.2)$$

via (2.9). Note that  $\widehat{\partial}^+ \varphi(\bar{x})$  reduces to the upper subdifferential (or superdifferential) of convex analysis if  $\varphi$  is concave. Furthermore, the subdifferential sets  $\widehat{\partial} \varphi(\bar{x})$  and  $\widehat{\partial}^+ \varphi(\bar{x})$  are nonempty simultaneously if and only if  $\varphi$  is Fréchet differentiable at  $\bar{x}$ .

As before, in the next theorem and in what follows the symbol  $\mathcal{L}$  stands for the collection of all the finite subsets of the natural series  $\mathbb{N}$ .

**Theorem 4.1 (upper subdifferential conditions for SIP with countable geometric constraints).** *Let  $\bar{x}$  be a local optimal solution to problem (4.1), where  $\varphi: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is an arbitrary extended-real-valued function finite at  $\bar{x}$ , and where the sets  $\Omega_i \subset \mathbb{R}^n$  for  $i \in \mathbb{N}$  are locally closed around  $\bar{x}$ . Assume that the system  $\{\Omega_i\}_{i \in \mathbb{N}}$  has the CHIP at  $\bar{x}$  and satisfies the NQC of Definition 3.10(b) at this point. Then we have the set inclusion*

$$-\widehat{\partial}^+ \varphi(\bar{x}) \subset \text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(\bar{x}; \Omega_i), I \in \mathcal{L} \right\}, \quad (4.3)$$

which reduces to that of

$$0 \in \nabla \varphi(\bar{x}) + \text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(\bar{x}; \Omega_i), I \in \mathcal{L} \right\}. \quad (4.4)$$

if  $\varphi$  is Fréchet differentiable at  $\bar{x}$ . If in addition the NCC of Definition 3.10(a) holds for  $\{\Omega_i\}_{i \in \mathbb{N}}$  at  $\bar{x}$ , then the closure operations can be omitted in (4.3) and (4.4).

**Proof.** It follows from [19, Proposition 5.2] that

$$-\widehat{\partial}^+ \varphi(\bar{x}) \subset \widehat{N} \left( \bar{x}; \bigcap_{i=1}^{\infty} \Omega_i \right). \quad (4.5)$$

Applying now to (4.5) the representation of Fréchet normals to countable set intersections from Theorem 3.12 under the assumed CHIP and NQC, we arrive at (4.3), where the closure operation can be omitted when the NCC holds at  $\bar{x}$ . If  $\varphi$  is Fréchet differentiable at  $\bar{x}$ , it follows that  $\widehat{\partial}^+ \varphi(\bar{x}) = \{\nabla \varphi(\bar{x})\}$ , and thus (4.3) reduces to (4.4).  $\square$

Note that the set inclusion (4.3) is trivial if  $\widehat{\partial}^+ \varphi(\bar{x}) = \emptyset$ , which is the case of, e.g., nonsmooth convex functions. On the other hand, the upper subdifferential necessary optimality condition (4.3) may be much more selective than its lower subdifferential counterparts when  $\widehat{\partial}^+ \varphi(\bar{x}) \neq \emptyset$ , which happens, in particular, for some remarkable classes of functions including concave, upper regular, semiconcave, upper- $C^1$ , and other ones important in various applications. The reader can find more information and comparison in [19, Subsection 5.1.1] and the commentaries therein concerning problems with finitely many geometric constraints.

Next let us present a lower subdifferential condition for the SIP problem (4.1) involving the basic subdifferential (2.8), which is nonempty for majority of nonsmooth functions; in particular, for any local Lipschitzian one. To formulate this condition, recall the notion of the *singular subdifferential* of  $\varphi$  at  $\bar{x}$  defined by

$$\partial^\infty \varphi(\bar{x}) := \{x^* \in \mathbb{R}^n \mid (x^*, 0) \in N((\bar{x}; \varphi(\bar{x})); \text{epi } \varphi)\}. \quad (4.6)$$

Note that  $\partial^\infty \varphi(\bar{x}) = \{0\}$  if  $\varphi$  is locally Lipschitzian around  $\bar{x}$ . Recall also that a set  $\Omega$  is *normally regular* at  $\bar{x}$  if  $N(\bar{x}; \Omega) = \widehat{N}(\bar{x}; \Omega)$ . This is the case, in particular, of locally convex and other “nice” sets; see, e.g., [18, 21] and the references therein.

**Theorem 4.2 (lower subdifferential subdifferential conditions for SIP with countable geometric constraints.)** *Let  $\bar{x}$  be a local optimal solution to problem (4.1) with a lower semicontinuous cost function  $\varphi: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  finite at  $\bar{x}$  and a countable system  $\{\Omega_i\}_{i \in \mathbb{N}}$  of sets locally*

closed around  $\bar{x}$ . Assume that the feasible solution set  $\Omega := \bigcap_{i=1}^{\infty} \Omega_i$  is normally regular at  $\bar{x}$ , that the system  $\{\Omega_i\}_{i \in \mathbb{N}}$  satisfies the CHIP (3.1) and the NQC (3.20) at  $\bar{x}$ , and that

$$\text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(\bar{x}; \Omega_i), I \in \mathcal{L} \right\} \cap (-\partial^\infty \varphi(\bar{x})) = \{0\}, \quad (4.7)$$

which holds, in particular, when  $\varphi$  is locally Lipschitzian around  $\bar{x}$ . Then we have

$$0 \in \partial \varphi(\bar{x}) + \text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(\bar{x}; \Omega_i), I \in \mathcal{L} \right\}. \quad (4.8)$$

The closure operations can be omitted in (4.7) and (4.8) if the NCC (3.19) is satisfied at  $\bar{x}$ .

**Proof.** It follows from [19, Proposition 5.3] that

$$0 \in \partial \varphi(\bar{x}) + N(\bar{x}; \Omega) \text{ provided that } \partial^\infty \varphi(\bar{x}) \cap (-N(\bar{x}; \Omega)) = \{0\} \quad (4.9)$$

for the optimal solution  $\bar{x}$  to the problem under consideration with the feasible solution set  $\Omega := \bigcap_{i=1}^{\infty} \Omega_i$ . Since the set  $\Omega$  is normally regular at  $\bar{x}$ , we can replace  $N(\bar{x}; \Omega)$  by  $\hat{N}(\bar{x}; \Omega)$  in (4.9). Applying now Theorem 3.12 to the countable set intersection  $\Omega$  in (4.9) under the assumptions made, we arrive at all the conclusions of this theorem.  $\square$

Next we consider a SIP problem with *countable operator constraints* defined by:

$$\text{minimize } \varphi(x) \text{ subject to } f(x) \in \Theta_i \text{ as } i \in \mathbb{N}, \quad (4.10)$$

where  $\varphi: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ ,  $\Theta_i \subset \mathbb{R}^m$  for  $i \in \mathbb{N}$ , and  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ . The following statements are consequences of Theorems 4.1 and 4.2, respectively.

**Corollary 4.3 (upper and lower subdifferential conditions for SIP with operator constraints).** *Let  $\bar{x}$  be a local optimal solution to (4.10), where the function  $\varphi: \mathbb{R}^n$  is finite at  $\bar{x}$ , where the mapping  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is strictly differentiable at  $\bar{x}$  with the surjective (full rank) derivative, and where the sets  $\Theta_i \subset \mathbb{R}^m$  as  $i \in \mathbb{N}$  are locally closed around  $f(\bar{x})$  while satisfying the CHIP (3.1) and NQC (3.20) conditions at this point. The following assertions holds:*

(i) *We have the upper subdifferential optimality condition:*

$$-\hat{\partial}^+ \varphi(\bar{x}) \subset \text{cl} \left\{ \sum_{i \in I} \nabla f(\bar{x})^* y_i^* \mid y_i^* \in N(f(\bar{x}); \Theta_i), I \in \mathcal{L} \right\}, \quad (4.11)$$

(ii) *If  $\varphi$  is lower semicontinuous around  $\bar{x}$  and*

$$\text{cl} \left\{ \sum_{i \in I} \nabla f(\bar{x})^* y_i^* \mid y_i^* \in N(f(\bar{x}); \Theta_i), I \in \mathcal{L} \right\} \cap (-\partial^\infty \varphi(\bar{x})) = \{0\}, \quad (4.12)$$

*then we have the inclusion*

$$0 \in \partial \varphi(\bar{x}) + \text{cl} \left\{ \sum_{i \in I} \nabla f(\bar{x})^* y_i^* \mid y_i^* \in N(f(\bar{x}); \Theta_i), I \in \mathcal{L} \right\}. \quad (4.13)$$

Furthermore, the closure operations can be omitted in (4.11)–(4.13) if the set system  $\{\Theta_i\}_{i \in \mathbb{N}}$  satisfies the NCC (3.19) at  $f(\bar{x})$ .



**Proof.** Observe that problem (4.10) can be equivalently rewritten in the geometric form (4.1) with  $\Omega_i := f^{-1}(\Theta_i)$ ,  $i \in \mathcal{I}$ . Then employing the well-known results on representing the tangent and normal cones in (2.1) and (2.2) to inverse images of sets under strict differentiable mappings with surjective derivatives (see, e.g., [18, Theorem 1.17] and [21, Exercise 6.7]), we have

$$T(\bar{x}; f^{-1}(\Theta)) = \nabla f(\bar{x})^{-1} T(f(\bar{x}); \Theta) \quad \text{and} \quad N(\bar{x}; f^{-1}(\Theta)) = \nabla f(\bar{x})^* N(f(\bar{x}); \Theta). \quad (4.14)$$

It follows from the surjectivity of  $\nabla f(\bar{x})$  that the CHIP and NQC for  $\{\Theta_i\}_{i \in \mathcal{I}}$  at  $f(\bar{x})$  are equivalent, respectively, to the CHIP and NQC of  $\{\Omega_i\}_{i \in \mathcal{I}}$  at  $\bar{x}$ ; see [18, Lemma 1.18]. This implies the equivalence between the qualification and optimality conditions (4.11)–(4.13) for problem (4.10) under the assumptions made and the corresponding conditions (4.3), (4.7), and (4.8) for problem (4.1) established in Theorems 4.1 and 4.2. To complete the proof of the corollary, it suffices to observe similarly to (4.14) that the assumed NCC for  $\{\Theta_i\}_{i \in \mathcal{I}}$  at  $f(\bar{x})$  is equivalent under the surjectivity of  $\nabla f(\bar{x})$  to the NCC (3.19) for the inverse images  $\{\Omega_i\}_{i \in \mathcal{I}}$  at  $\bar{x}$ . Thus the possibility to omit the closure operations in the framework of the corollary follows directly from the corresponding statements of Theorems 4.1 and 4.2.  $\square$

The rest of this section concerns SIP problems with *countable inequality constraints*:

$$\text{minimize } \varphi(x) \quad \text{subject to } \varphi_i(x) \leq 0 \quad \text{as } i \in \mathcal{I}, \quad (4.15)$$

where the cost function  $\varphi$  is as in problems (4.1) and (4.10) while the constraints functions  $\varphi_i: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ ,  $i \in \mathcal{I}$ , are lower semicontinuous around the reference optimal solution. Note that problems with infinite inequality constraints are considered in the vast majority of publications on semi-infinite programming, where the main attention is paid to the case of convex or linear infinite inequalities; see below some comparison with known results for SIP of the latter types.

Although our methods are applied to problems (4.15) of the general inequality type, for simplicity and brevity we focus here on the case when the constraint functions  $\varphi_i$ ,  $i \in \mathcal{I}$ , are locally Lipschitzian around the optimal solution. In the general case we need to involve the singular subdifferential (4.6) of these functions; see the proofs below. Let us first introduce subdifferential counterparts of the normal qualification and closedness conditions from Definition 3.10.

**Definition 4.4 (subdifferential closedness and qualification conditions for countable inequality constraints).** Consider a countable constraint system  $\{\Omega_i\}_{i \in \mathcal{I}} \subset \mathbb{R}^n$  with

$$\Omega_i := \{x \in \mathbb{R}^n \mid \varphi_i(x) \leq 0\}, \quad i \in \mathcal{I}, \quad (4.16)$$

where the functions  $\varphi_i$  are locally Lipschitzian around  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$ . We say that:

(a) The system  $\{\Omega_i\}_{i \in \mathcal{I}}$  in (4.16) satisfies the SUBDIFFERENTIAL CLOSEDNESS CONDITION (SCC) at  $\bar{x}$  if the set

$$\left\{ \sum_{i \in \mathcal{I}} \lambda_i \partial \varphi_i(\bar{x}) \mid \lambda_i \geq 0, \lambda_i \varphi_i(\bar{x}) = 0, I \in \mathcal{L} \right\} \quad \text{is closed in } \mathbb{R}^n. \quad (4.17)$$

(b) The system  $\{\Omega_i\}_{i \in \mathcal{I}}$  in (4.16) satisfies the SUBDIFFERENTIAL QUALIFICATION CONDITION (SQC) at  $\bar{x}$  if the following implication holds:

$$\left[ \sum_{i=1}^{\infty} \lambda_i x_i^* = 0, x_i^* \in \partial \varphi_i(\bar{x}), \lambda_i \geq 0, \lambda_i \varphi_i(\bar{x}) = 0 \right] \implies [\lambda_i = 0 \text{ for all } i \in \mathcal{I}]. \quad (4.18)$$

The next theorem provides necessary optimality conditions of both upper and lower subdifferential types for SIP problems (4.15) without any smoothness and/or convexity assumptions.

**Theorem 4.5 (upper and lower subdifferential conditions for general SIP with inequality constraints).** *Let  $\bar{x}$  be a local optimal solution to problem (4.15), where the constraint functions  $\varphi_i: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  are locally Lipschitzian around  $\bar{x}$  for all  $i \in \mathbb{N}$ . Assume that the level set system  $\{\Omega_i\}_{i \in \mathbb{N}}$  in (4.16) has the CHIP at  $\bar{x}$  and that the SQC (4.18) is satisfied at this point. Then the following assertions hold:*

(i) *We have the upper subdifferential optimality condition:*

$$-\widehat{\partial}^+ \varphi(\bar{x}) \subset \text{cl} \left\{ \sum_{i \in I} \lambda_i \partial \varphi_i(\bar{x}) \mid \lambda_i \geq 0, \lambda_i \varphi_i(\bar{x}) = 0, I \in \mathcal{L} \right\}, \quad (4.19)$$

where the closure operation can be omitted if the SCC (4.17) is satisfied at  $\bar{x}$ .

(ii) *Assume in addition that  $\varphi$  is lower semicontinuous around  $\bar{x}$  and that*

$$\text{cl} \left\{ \sum_{i \in I} \lambda_i \partial \varphi_i(\bar{x}) \mid \lambda_i \geq 0, \lambda_i \varphi_i(\bar{x}) = 0, I \in \mathcal{L} \right\} \cap (-\partial^\infty \varphi(\bar{x})) = \{0\}, \quad (4.20)$$

which is automatic if  $\varphi$  is locally Lipschitzian around  $\bar{x}$ . Then

$$0 \in \partial \varphi(\bar{x}) + \text{cl} \left\{ \sum_{i \in I} \lambda_i \partial \varphi_i(\bar{x}) \mid \lambda_i \geq 0, \lambda_i \varphi_i(\bar{x}) = 0, I \in \mathcal{L} \right\} \quad (4.21)$$

with removing the closure operation in (4.20) and (4.21) when the SCC (4.17) holds at  $\bar{x}$ .

**Proof.** It is well known from the calculus of basic normals and subgradients that

$$N(\bar{x}; \Omega) \subset \mathbb{R}_+ \partial \vartheta(\bar{x}) := \{ \lambda x^* \in \mathbb{R}^n \mid x^* \in \partial \vartheta(\bar{x}), \lambda \geq 0 \} \text{ for } \Omega := \{ x \in \mathbb{R}^n \mid \vartheta(x) \leq 0 \} \quad (4.22)$$

provided that  $\vartheta: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is locally Lipschitzian around  $\bar{x}$ ; see, e.g., [18, Theorem 3.86]. Now we apply inclusion (4.22) to each set  $\Omega_i$  in (4.16) and substitute this into the NQC (3.20) as well as into the qualification condition (4.7) and the optimality conditions (4.3) and (4.8) for problem (4.1) with the constraint sets (4.16). It follows in this way that the SQC (4.18) and all the relationships (4.19)–(4.21) imply the aforementioned conditions of Theorems 4.1 and (4.2) in the setting (4.15) under consideration. It shows furthermore that the SCC (4.17) yields the NCC (3.19) for the sets  $\Omega_i$  in (4.16), which thus completes the proof of the theorem.  $\square$

Now we consider in more detail the case of *convex* constraint functions  $\varphi_i$  in (4.15). Note that the validity of the SQC (4.18) is ensured in the case by the interior-type condition (3.21) of Proposition 3.11. The next theorem justifies necessary optimality conditions for problems with countable convex inequalities, which does not require either interiority-type or SQC constraint qualifications while containing a qualification condition that implies both the CHIP and SCC in (4.17). Let us first recall this condition; see [12, 13] and the references therein. We say that the SIP problem (4.15) with the constraints given by convex functions  $\varphi_i$ ,  $i \in \mathbb{N}$ , satisfies the *Farkas-Minkowski constraint qualification* (FMCQ) if the set

$$\text{co} \left[ \text{cone} \bigcup_{i=1}^{\infty} \text{epi} \varphi_i^* \right] \text{ is closed in } \mathbb{R}^n \times \mathbb{R}, \quad (4.23)$$

where  $\vartheta^*(x^*) := \sup \{ \langle x^*, x \rangle - \vartheta(x) \mid x \in \mathbb{R}^n \}$  stands for the conjugate function to  $\vartheta: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ .

**Theorem 4.6 (upper and lower subdifferential conditions for SIP with convex inequality constraints).** *Let all the general assumptions but SQC (4.18) of Theorem 4.5 be fulfilled at the local optimal solution  $\bar{x}$  to (4.15). Assume also that the constraint functions  $\varphi_i$ ,  $i \in \mathbb{N}$ , are convex. The both assertion (i) and (ii) of Theorem 4.5 are satisfied. Furthermore, the fulfillment of the FMCQ (4.23) implies that the CHIP (3.1) holds automatically and that the closure operation in (4.19)–(4.21) can be omitted.*

**Proof.** Note first that inclusion (4.22) holds as equality for convex functions, i.e.,

$$N(\bar{x}; \Omega_i) = \mathbb{R}_+ \partial \varphi_i(\bar{x}) \text{ for } \Omega_i = \{x \in \mathbb{R}^n \mid \varphi_i(x) \leq 0\}, \quad i \in \mathbb{N}. \quad (4.24)$$

Combining (4.24) with Theorem 3.2 and taking into account that  $N(\bar{x}; \Omega_i) = \{0\}$  when  $\varphi_i(\bar{x}) < 0$ , we can equivalently rewrite the assumed CHIP in the form

$$N\left(\bar{x}; \bigcap_{i=1}^{\infty} \Omega_i\right) = \text{cl co} \bigcup_{i \in J(\bar{x})} [\mathbb{R}_+ \partial \varphi_i(\bar{x})] \text{ with } J(\bar{x}) := \{i \in \mathbb{N} \mid \varphi_i(\bar{x}) = 0\}. \quad (4.25)$$

Substituting the latter into the upper and lower subdifferential optimality conditions

$$-\hat{\partial}^+ \varphi(\bar{x}) \subset N\left(\bar{x}; \bigcap_{i=1}^{\infty} \Omega_i\right) \text{ and } 0 \in \partial \varphi(\bar{x}) + N\left(\bar{x}; \bigcap_{i=1}^{\infty} \Omega_i\right)$$

for problem (4.15), which follow from [19, Propositions 5.2 and 5.3], respectively, we arrive at the conclusions in (i) and (ii) of Theorem 4.5.

To complete the proof of the theorem, it remains to check that the FMCQ (4.23) simultaneously implies the fulfillments of the CHIP (3.1) and the SCC (4.17). It follows from [12, Corollary 3.6] that the FMCQ yields the representation

$$N\left(\bar{x}; \bigcap_{i=1}^{\infty} \Omega_i\right) = \bigcup_{\lambda \in A(\bar{x})} \left[ \sum_{i \in J(\bar{x})} \lambda_i \partial \varphi_i(\bar{x}) \right] \quad (4.26)$$

for the constraint sets  $\Omega_i$ , where  $A(\bar{x})$  denotes the collection of Lagrange multipliers  $\lambda = (\lambda_i)_{i \in \mathbb{N}}$  such that  $\lambda \in A(\bar{x})$  if and only if  $\lambda_i \geq 0$  for  $i \in J(\bar{x})$  and  $\lambda_i = 0$  otherwise. We obviously have from (4.24) and (4.26) that

$$N\left(\bar{x}; \bigcap_{i=1}^{\infty} \Omega_i\right) = \text{co} \bigcup_{i \in J(\bar{x})} \mathbb{R}_+ \partial \varphi_i(\bar{x}) = \text{co} \bigcup_{i=1}^{\infty} N(\bar{x}; \Omega_i). \quad (4.27)$$

Since the normal cone  $N(\bar{x}; \Omega)$  is closed, it follows from (4.27) that the set  $\text{co}\{\bigcup_{i \in J(\bar{x})} [\mathbb{R}_+ \partial \varphi_i(\bar{x})]\}$  is closed as well; the latter is clearly equivalent to the SCC (4.17) at  $\bar{x}$ . On the other hand, we have from (4.27) that the strong CHIP (3.3) holds, which implies the fulfillment of the CHIP (3.1) by Theorem 3.2 and thus completes the proof of this theorem.  $\square$

Next we present efficient specifications of both upper and lower subdifferential optimality conditions from Theorem 4.6 for SIP with linear inequality constraints. In the finite-dimensional countable case under consideration the results obtained in this way reduce to those from [8, Theorems 3.1 and 4.1] while it is *not* assumed here the strong Slater condition and the coefficient boundedness imposed in [8]. For simplicity we consider the case of homogeneous constraints and suppose that  $\bar{x} = 0$  is a local optimal solution.

**Corollary 4.7 (upper and lower subdifferential conditions for SIP with linear inequality constraints).** *Let  $\bar{x} = 0$  be a local optimal solution to the SIP problem*

$$\text{minimize } \varphi(x) \text{ subject to } \langle a_i, x \rangle \leq 0 \text{ for all } i \in \mathcal{N}, \quad (4.28)$$

where  $\varphi: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is finite at the origin. Then we have the inclusions

$$-\widehat{\partial}^+ \varphi(0) \subset \text{cl co} \left[ \bigcup_{i=1}^{\infty} \{ \lambda a_i \mid \lambda \geq 0 \} \right]. \quad (4.29)$$

$$0 \in \partial \varphi(0) + \text{cl co} \left[ \bigcup_{i=1}^{\infty} \{ \lambda a_i \mid \lambda \geq 0 \} \right], \quad (4.30)$$

where (4.30) holds provided that  $\varphi$  is lower semicontinuous around the origin and

$$\left( \text{cl co} \left[ \bigcup_{i=1}^{\infty} \{ \lambda a_i \mid \lambda \geq 0 \} \right] \right) \cap (-\partial^\infty \varphi(0)) = \{0\}. \quad (4.31)$$

Furthermore, the FMCQ implies that the closure operations can be omitted in (4.29)–(4.31).

**Proof.** Since the CHIP is automatic for the linear inequality system in (4.28) at the origin and by Corollary 3.3 we have the normal cone representation (3.10), all the results of this corollary follow from the corresponding results of Theorem 4.6.  $\square$

Finally in this section, we present several examples illustrating the qualification conditions imposed in Theorem 4.6 and their comparison with known results in the literature.

**Example 4.8 (comparison of qualification conditions).** All the examples below concern lower subdifferential conditions for SIP problems (4.15) with convex cost and constraint functions.

**(i) The CHIP (3.1) and the SCC (4.17) are independent.** Consider a linear constraint system in (4.7) at  $\bar{x} = 0 \in \mathbb{R}^2$  for  $\varphi_i(x) = \langle a_i, x \rangle$  with  $a_i = (1, i)$  as  $i \in \mathcal{N}$ , which has the CHIP. At the same time the set

$$\text{co} \bigcup_{i=0}^{\infty} \mathbb{R}_+ \partial \varphi_i(\bar{x}) = \text{co} \{ \lambda(1, i) \in \mathbb{R}^2 \mid \lambda \geq 0, i \in \mathcal{N} \} = \mathbb{R}_+^2 \setminus \{ (0, \lambda) \mid \lambda > 0 \}$$

is not closed, and hence the SCC (4.17) does not hold. On the other hand, for the quadratic functions  $\varphi_i(x) = ix_i^2 - x_2$  as  $i \in \mathcal{N}$  as  $x = (x_1, x_2) \in \mathbb{R}^2$ , we get  $\partial \varphi_i(0) = \nabla \varphi_i(0) = (0, -1)$ , and hence the SCC (4.17) holds at the origin while the CHIP is violated at this point by Example 3.4(ii).

**(ii) (CHIP and SCC versus FMCQ and CQC).** Besides the FMCQ (4.23), another qualification condition is employed in [12, 13] to obtain necessary optimality conditions of *Karush-Kuhn-Tucker* (KKT) type (no closure operation in (4.21)) for *fully convex* SIP problems (4.15) involving all the convex functions  $\varphi$  and  $\varphi_i$ . This condition, named the *closedness qualification condition* (CQC) is formulated as follows via the convex conjugate functions: the set

$$\text{epi } \varphi^* + \text{co} \left[ \text{cone} \bigcup_{i=1}^{\infty} \text{epi } \varphi_i^* \right] \text{ is closed in } \mathbb{R}^n \times \mathbb{R}. \quad (4.32)$$

It is obvious that the FMCQ implies the CQC while the latter is implied only for fully convex SIP problems. The next example presents a fully convex SIP problem satisfying both CHIP and

SCC but not the CQC (and hence not FMCQ). This shows that Theorem 4.6 holds in this case to produce the KKT optimality condition while the corresponding result of [12] is not applicable.

Consider the SIP (4.6) with  $x = (x_1, x_2) \in \mathbb{R}^2$ ,  $\bar{x} = (0, 0)$ ,  $\varphi(x) = -x_2$ , and

$$\varphi_i(x_1, x_2) = \begin{cases} ix_1^2 - x_2 & \text{if } x_1 < 0, \\ -x_2 & \text{if } x_1 \geq 0, \end{cases} \quad i \in \mathbb{N}.$$

We have  $\partial\varphi_i(\bar{x}) = \nabla\varphi_i(\bar{x}) = (0, -1)$  for all  $i \in \mathbb{N}$ , and hence the SCC (4.17) holds. It is easy to check that the CHIP holds at  $\bar{x}$ , since

$$T\left(\bar{x}; \bigcap_{i=1}^{\infty} \Omega_i\right) = T(\bar{x}; \Omega_i) = \mathbb{R} \times \mathbb{R}_+ \text{ for } \Omega_i := \{x \in \mathbb{R}^2 \mid \varphi_i(x) \leq 0\}, \quad i \in \mathbb{N}.$$

On the other hand, for  $x^* = (\lambda_1, \lambda_2) \in \mathbb{R}^n$  we compute the conjugate functions by

$$\varphi^*(x^*) = \begin{cases} 0 & \text{if } (\lambda_1, \lambda_2) = (0, -1), \\ \infty & \text{otherwise} \end{cases} \quad \text{and} \quad \varphi_i^*(x^*) = \begin{cases} \frac{\lambda_1^2}{4i} & \text{if } \lambda_1 \leq 0, \lambda_2 = -1, \\ \infty & \text{otherwise.} \end{cases}$$

This shows that the convex sets

$$\text{co} \left[ \text{cone} \bigcup_{i=0}^{\infty} \text{epi } \varphi_i^* \right] \quad \text{and} \quad \text{epi } \varphi^* + \text{co} \left[ \text{cone} \bigcup_{i=0}^{\infty} \text{epi } \varphi_i^* \right]$$

are not closed in  $\mathbb{R}^2 \times \mathbb{R}$ , and hence the FMCQ (4.23) and the CQC (4.32) are not satisfied.

## 5 Applications to Multiobjective Optimization

The last section of this paper concerns problems of multiobjective optimization with set-valued objectives and countable constraints. Although optimization problems with single-valued/vector and (to a lesser extent) set-valued objectives have been widely considered in optimization and equilibrium theories as well as in their numerous applications (see, e.g., the books [11, 16, 19] and the references therein), we are not familiar with the study of such problems involving countable constraints. Our interest is devoted to deriving necessary optimality conditions for problems of this type based on the dual-space approach to the general multiobjective optimization theory developed in [2, 3, 19] and the new tangential extremal principle established in [20].

The main problem of our consideration is as follows:

$$\text{minimize } F(x) \text{ subject to } x \in \Omega := \bigcap_{i=1}^{\infty} \Omega_i \subset \mathbb{R}^n, \quad (5.1)$$

where  $\Omega_i$ ,  $i \in \mathbb{N}$ , are closed subsets of  $\mathbb{R}^n$ , where  $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  is a set-valued mapping of closed graph, and where “minimization” is understood with respect to some partial ordering “ $\leq$ ” on  $\mathbb{R}^m$ . We pay the main attention to the multiobjective problems with the *Pareto-type* ordering:

$$y_1 \leq y_2 \text{ if and only if } y_2 - y_1 \in \Theta,$$

where  $\emptyset \neq \Theta \subset \mathbb{R}^m \setminus \{0\}$  is a closed, convex, and pointed ordering cone. In the aforementioned references the reader can find more discussions on this and other ordering relations.

Recall that a point  $(\bar{x}, \bar{y}) \in \text{gph } F$  with  $\bar{x} \in \Omega$  is a *local minimizer* of problem (5.1) if there exists a neighborhood  $U$  of  $\bar{x}$  such that there is no  $y \in F(\Omega \cap U)$  preferred to  $\bar{y}$ , i.e.,

$$F(\Omega \cap U) \cap (\bar{y} - \Theta) = \{\bar{y}\}. \quad (5.2)$$

Note that notion (5.2) does not take into account the image localization of minimizers around  $\bar{y} \in F(\bar{x})$ , which is essential for certain applications of set-valued minimization, e.g., to economic modeling; see [3]. A more appropriate notion for such problems is defined in [3] under the name of *fully localized minimizers* as follows: there are neighborhoods  $U$  of  $\bar{x}$  and  $V$  of  $\bar{y}$  such that

$$F(\Omega \cap U) \cap (\bar{y} - \Theta) \cap V = \{\bar{y}\}. \quad (5.3)$$

The next result establishes necessary optimality conditions of the coderivative type for fully localized minimizers of problem (5.1) with countable constraints based on the approach of [19] to problems of multiobjective optimizations, its implementations in [2, 3] specifically for problems with set-valued criteria, and the tangential extremal principle for countable sets [20]. We address here fully localized minimizers for multiobjective problems (5.1) with normally regular feasible sets, i.e., when  $N(\bar{x}; \Omega) = \hat{N}(\bar{x}; \Omega)$ , which particularly includes the case of convex set  $\Omega_i$ ,  $i \in \mathbb{N}$ .

**Theorem 5.1 (optimality conditions for fully localized minimizers of multiobjective problems with countable constraints and normally regular feasible sets).** *Let the pair  $(\bar{x}, \bar{y}) \in \text{gph } F$  be a fully localized minimizer for (5.1) with the CHIP system of countable constraints  $\{\Omega_i\}_{i \in \mathbb{N}}$ . Assume that the feasible set  $\Omega = \bigcap_{i=1}^{\infty} \Omega_i$  is normally regular at  $\bar{x} \in \Omega$  and that the NQC (3.20) and the coderivative qualification condition*

$$D^*F(\bar{x}, \bar{y})(0) \bigcap \left[ -\text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(\bar{x}; \Omega_i), I \in \mathcal{L} \right\} \right] = \{0\} \quad (5.4)$$

*are satisfied. Then there is  $0 \neq y^* \in -N(0; \Theta)$  such that*

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + \text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(\bar{x}; \Omega_i), I \in \mathcal{L} \right\}. \quad (5.5)$$

**Proof.** Applying [3, Theorem 3.4] for fully localized minimizers of set-valued optimization problems with abstract geometric constraints  $x \in \Omega$  (cf. also [2, Theorem 5.3] for the case of local minimizers (5.2) and [19, Theorem 5.59] for vector single-objective counterparts), we find

$$0 \neq -y^* \in N(0; \Theta) \quad \text{and} \quad x^* \in D^*F(\bar{x}, \bar{y})(y^*) \cap (-N(\bar{x}; \Omega)) \quad (5.6)$$

provided the fulfillment of the qualification condition

$$D^*F(\bar{x}, \bar{y})(0) \cap (-N(\bar{x}; \Omega)) = \{0\}. \quad (5.7)$$

To complete the proof of the theorem, it suffices to employ in (5.6) and (5.7) the sum rule for countable set intersections from Theorem 3.12 by taking into account the assumed normal regularity of the intersection set  $\Omega$  at  $\bar{x}$ .  $\square$

Note that the qualification condition (5.4) holds automatically if the objective mapping  $F$  is *Lipschitz-like* (or has the Aubin property) around  $(\bar{x}, \bar{y}) \in \text{gph } F$ , i.e., there are neighborhoods  $U$  of  $\bar{x}$  and  $V$  of  $\bar{y}$  such that

$$F(x) \cap V \subset F(u) + \ell \|x - u\| B \quad \text{for all } x, u \in U$$

with some number  $\ell \geq 0$ . Indeed, it follows from the Mordukhovich criterion in [21, Theorem 9.40] (see also [18, Theorem 4.10] and the references therein) that  $D^*F(\bar{x}, \bar{y})(0) = \{0\}$  in this case.

Next we introduce two kinds of “graphical” minimizers for multiobjective problems for which, in particular, we can avoid the normal regularity assumption in optimality conditions of type (5.5) in Theorem 5.1. The definition below concerns multiobjective optimization problems with general geometric constraints that may not be represented as countable set intersections.

**Definition 5.2 (graphical and tangential graphical minimizers).** *Let  $(\bar{x}, \bar{y}) \in \text{gph } F$  with  $\bar{x} \in \Omega$ . We say that:*

(i)  *$(\bar{x}, \bar{y})$  is a LOCAL GRAPHICAL MINIMIZER to problem (5.1) if there are neighborhoods  $U$  of  $\bar{x}$  and  $V$  of  $\bar{y}$  such that*

$$\text{gph } F \cap [\Omega \times (\bar{y} - \Theta)] \cap (U \times V) = \{(\bar{x}, \bar{y})\}. \quad (5.8)$$

(ii)  *$(\bar{x}, \bar{y})$  is a LOCAL TANGENTIAL GRAPHICAL MINIMIZER to problem (5.1) if*

$$T((\bar{x}, \bar{y}); \text{gph } F) \cap [T(\bar{x}; \Omega) \times (-\Theta)] = \{0\}. \quad (5.9)$$

Similarly to the discussions and examples on relationships between local extremal and tangentially extremal points of set systems given in [20], we observe that the optimality notions in Definition 5.2 are *independent* of each other. Let us now compare the graphical optimality of Definition 5.2(i) with fully localized minimizers of (5.3).

**Proposition 5.3 (relationships between fully localized and graphical minimizers).** *Let  $(\bar{x}, \bar{y}) \in \text{gph } F$  be a feasible solution to problem (5.1) with general geometric constraints. Then the following assertions are satisfied:*

(i)  *$(\bar{x}, \bar{y})$  is a local graphical minimizer if it is a fully localized minimizer for this problem.*

(ii) *The opposite implication holds if there is a neighborhood  $U$  of  $\bar{x}$  such that  $\bar{y} \notin F(x)$  for every  $\bar{x} \neq x \in \Omega \cap U$ .*

**Proof.** To justify (i), assume that  $(\bar{x}, \bar{y})$  is a local graphical minimizer, take its neighborhood  $U \times V$  from Definition 5.2(i), and pick any

$$y \in F(\Omega \cap U) \cap (\bar{y} - \Theta) \cap V.$$

Then there is  $x \in \Omega \cap U$  such that  $y \in F(x)$ , and so

$$(x, y) \in \text{gph } F \cap [\Omega \times (\bar{y} - \Theta)] \cap (U \times V) = \{(\bar{x}, \bar{y})\}.$$

Thus  $F(\Omega \cap U) \cap (\bar{y} - \Theta) \cap V = \{\bar{y}\}$ , i.e.,  $(\bar{x}, \bar{y})$  is a fully localized minimizer for (5.1).

Next we prove (ii). Suppose that  $(\bar{x}, \bar{y})$  is a fully localized minimizer with a neighborhood  $U \times V$ , shrink  $U$  so that the assumption in (ii) holds, and take

$$(x, y) \in \text{gph } F \cap [\Omega \times (\bar{y} - \Theta)] \cap (U \times V).$$

Since  $y \in F(x)$ , it follows that  $y \in F(\Omega \cap U) \cap (\bar{y} - \Theta) \cap V = \{\bar{y}\}$ . If  $x \neq \bar{x}$ , the latter contradicts the assumption in (ii). Thus  $x = \bar{x}$ , which completes the proof of the proposition.  $\square$

The next theorem uses the full strength of the tangential extremal principle of [20] justifying the necessary optimality conditions of Theorem 5.1 for *tangential* graphical minimizers of the multiobjective problem (5.1) with countable constraints without imposing the normal regularity requirement of the feasible set.

**Theorem 5.4 (optimality conditions for tangential graphical minimizers).** *Let  $(\bar{x}, \bar{y})$  be a local tangential graphical minimizer for problem (5.1) under the fulfillment all the assumptions of Theorem 5.1 but the normal regularity of  $\Omega$  at  $\bar{x}$ . Suppose in addition that  $\text{int}\Theta \neq \emptyset$ . Then there is  $0 \neq y^* \in -N(0; \Theta)$  such that the necessary optimality condition (5.5) is satisfied.*

**Proof.** We have by Definition 5.2(ii) that  $T((\bar{x}, \bar{z}); \text{gph } F) \cap [\Lambda \times (-\Theta)] = \{0\}$  with  $\Lambda := T(\bar{x}; \Omega)$ . Since the system  $\{\Omega_i\}_{i \in \mathbb{N}}$  has the CHIP at  $\bar{x}$ , it follows that

$$\Lambda = \bigcap_{i=1}^{\infty} \Lambda_i \quad \text{with} \quad \Lambda_i := T(\bar{x}; \Omega_i).$$

Further, define the closed cones  $\Gamma_0 := T((\bar{x}, \bar{z}); \text{gph } F)$  and  $\Gamma_i := \Lambda_i \times (-\Theta)$  as  $i \in \mathbb{N}$  with  $\bigcap_{i=0}^{\infty} \Gamma_i = \{0\}$  and show that for any  $\xi \in \Theta$  we get

$$\bigcap_{i=1}^{\infty} \Gamma_i \cap [\Gamma_0 + (0, \xi)] = \emptyset. \quad (5.10)$$

Indeed, supposing the contrary gives us a vector  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$  with  $(x, y - \xi) \in \Gamma_0$  and  $(x, y) \in \Lambda_i \times (-\Theta)$  for all  $i \in \mathbb{N}$ . Since  $\Theta$  is a closed and convex cone, we also have the inclusion  $(x, y - \xi) \in \Lambda_i \times (-\Theta) = \Gamma_i$  as  $i \in \mathbb{N}$ , and hence

$$(x, y - \xi) \in \bigcap_{i=0}^{\infty} \Gamma_i = \{0\}.$$

It follows therefore that  $y = \xi \in -\Theta$ , which implies by the pointedness of the cone  $\Theta$  that  $\xi \in (-\Theta) \cap \Theta = \{0\}$ , a contradiction justifying (5.10).

The latter means that  $\{\Gamma_i\}, i = 0, 1, \dots$ , is a countable system of cones extremal at the origin with the nonoverlapping condition  $\bigcap_{i=0}^{\infty} \Gamma_i = \{0\}$ . Now applying the tangential extremal principle of Theorem 2.1 to this system of cones and using also [20, Proposition 2.1], we get elements  $(x_i^*, y_i^*)$  as  $i = 0, 1, \dots$  satisfying the relationships

$$(x_0^*, y_0^*) \in N(0; \Gamma_0) \subset N((\bar{x}, \bar{y}); \text{gph } F), \quad (5.11)$$

$$(x_i^*, y_i^*) \in N(0; \Gamma_i) \subset N(\bar{x}; \Omega_i) \times [-N(0; \Theta)], \quad i \in \mathbb{N}, \quad (5.12)$$

$$\sum_{i=0}^{\infty} \frac{1}{2^i} (x_i^*, y_i^*) = 0, \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{1}{2^i} (\|x_i^*\|^2 + \|y_i^*\|^2) = 1. \quad (5.13)$$

It follows from (5.11)–(5.13) that

$$x_0^* \in D^*F(\bar{x}, \bar{y})(-y_0^*) \quad \text{and} \quad -y_0^* = \sum_{i=1}^{\infty} \frac{1}{2^i} y_i^* \in -N(0; \Theta), \quad (5.14)$$

where the latter inclusion holds by the convexity and closedness of the cone  $N(0; \Theta)$ .

There are the two possible cases in (5.14):  $y_0^* \neq 0$  and  $y_0^* = 0$ . In the first case we get

$$0 \in D^*F(\bar{x}, \bar{y})(-y_0^*) + \sum_{i=1}^{\infty} \frac{1}{2^i} x_i^*,$$



which readily implies the optimality condition (5.5) with  $0 \neq y^* := -y_0^* \in -N(0; \Theta)$ ; cf. the proof of the second part of [20, Theorem 5.4].

To complete the proof of this theorem, it remains to show that the case of  $y_0^* = 0$  in (5.14) cannot be realized under the imposed qualification conditions (3.20) and (5.4). Indeed, for  $y_0^* = 0$  we have from (5.12) and (5.14) that

$$-\frac{1}{2}y_1^* = \sum_{i=2}^{\infty} \frac{1}{2^i} y_i^* \in [-N(0; \Theta)] \cap N(0; \Theta). \quad (5.15)$$

Since the cone  $\Theta$  is convex, it follows from (5.15) that

$$\langle y_1^*, y \rangle \leq 0 \quad \text{and} \quad \langle y_1^*, y \rangle \geq 0 \quad \text{for any } y \in \Theta,$$

i.e.,  $\langle y_1^*, y \rangle = 0$  on  $\Theta$ . The latter implies that  $y_1^* = 0$  by  $\text{int}\Theta \neq \emptyset$ .

Proceeding in this way by induction gives us that  $y_i^* = 0$  for all  $i \in \mathbb{N}$ . Now it follows from (5.12) and the first inclusion in (5.14) that  $x_0^* = 0$  by the assumed coderivative qualification condition (5.4). Hence we get from (5.13) the relationships

$$\sum_{i=0}^{\infty} \frac{1}{2^i} x_i^* = 0 \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{1}{2^i} \|x_i^*\|^2 = 1,$$

which contradict the assumed NQC (3.20) and thus complete the proof of the theorem.  $\square$

Note in conclusion that, similarly to Section 4, we can develop necessary optimality conditions for multiobjective problems with countable constraints of operation and inequality types.

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